Change of Variables

To simplify Eq. (11), we introduce the following coordinate transformation for the dependent variable $x(t)$:

$$x(t) = Z(t)r(t) \tag{12}$$

where $Z(t)$ is given by Eq. (10) and $r(t)$ is a vector function which is to be determined.

Upon differentiating Eq. (12) we find

$$\dot{x} = \dot{Z}r + Z\dot{r} \tag{13}$$

or

$$\dot{x} = [\bar{A}Z + Z\bar{A}^T - BR^{-1}B^T]r + Z\dot{r} \tag{14}$$

where $\dot{Z}$ in Eq. (13) has been replaced by the right side of Eq. (9).

The differential equation for $r(t)$ is obtained by introducing Eqs. (12) and (14) into Eq. (11), leading to

$$Z(\dot{r} + \bar{A}^T r) = 0 \tag{15}$$

from which it follows that the linear constant coefficient vector differential equation for $r(t)$ is given by

$$\dot{r} = -\bar{A}^T r; \quad r_0 = Z^{-1}(t_0)x_0 \tag{16}$$

where the solution for $r(t)$ follows as

$$r(t) = e^{-\bar{A}^T(t-t_0)} r_0 \tag{17}$$

Substituting Eq. (17) into Eq. (12) produces the desired solution for the state trajectories as

$$x(t) = \Phi(t,t_0)x_0 \tag{18}$$

where $\Phi(t,t_0) = Z(t)e^{-\bar{A}^T(t-t_0)}Z^{-1}(t_0)$ is the system state transition matrix, and $\Phi(t,t_0)$ satisfies the following matrix differential equation

$$\dot{\Phi}(t,t_0) = [A - BR^{-1}B^T P(t)]\Phi(t,t_0); \quad \Phi(t_0,t_0) = I$$

Recursion Relationship for Evaluating the State at Discrete Times

If the solution for $x(t)$ is required at the discrete times $t_k = t_0 + k\Delta t$ $(k = 1,...,N)$ for $\Delta t = (t_f - t_0)/N$, then Eq. (18) can be written as

$$x(t_k) = Z_{ss}a_k + b_k \qquad (k = 0,...,N) \tag{19}$$

where

$$a_0 = r_0$$

$$b_0 = e^{\bar{A}(t_0 - t_f)}[Z(t_f) - Z_{ss}]e^{\bar{A}^T(t_0 - t_f)} r_0$$

$$a_k = e^{-\bar{A}^T \Delta t} a_{k-1}$$

$$b_k = e^{\bar{A}\Delta t} b_{k-1}$$

Conclusions

A straightforward algorithm has been presented for generating the state trajectories for a feedback control system. The algorithm is computationally efficient in that no numerical integration is required and simple recursion relationships generate the desired solution at discrete times. Furthermore, this algorithm has signficant potential if used in conjunction with algorithms which attempt to enhance system robustness by iteratively refining the weighting matrices appearing in the performance index.

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Isochrones for Maximum Endurance Horizontal Gliding Flight

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Nomenclature

C_1,C_2,C_3	= constants of integration
C_D	= drag coefficient, $C_{D0} + KC_L^2$
C_{D0}	= zero lift drag coefficient, constant
C_L	= lift coefficient
C_L^*	= C_L for maximum lift-to-drag ratio
D	= drag
E^*	= maximum lift-to-drag ratio
g	= gravitational acceleration
H	= Hamiltonian function
K	= induced drag factor, constant
L	= lift
n	= load factor, $L/W = 1/\cos\mu$
p_x,p_y,p_u,p_ψ	= adjoint variables

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S	= reference area
t	= time
u	= dimensionless speed, V/V_0
V	= vehicle speed
W	= vehicle weight
x,y	= dimensionless coordinates, $x=gX/V_0^2$, $y=gY/V_0^2$
X, Y	= vehicle position coordinates
θ	= dimensionless time, gt/V_0
λ	= normalized lift coefficient, C_L/C_L^*
μ	= bank angle
ρ	= atmosphere density
ψ	= velocity yaw angle
ω	= dimensionless wing loading, $2W/\rho S V_0^2 C_L^*$

Subscripts

f	= final value
max	= maximum value
0	= initial value
t	= target

Introduction

THE problem of maximum endurance gliding flight in a horizontal plane, exemplified by a small lifting vehicle capable of carrying a weapon system and gliding in a horizontal plane at a certain low altitude, has been investigated in Refs. 1-4. The most general form of this problem is to specify the target position within the reachable zone[4] and then to solve for the maximum endurance trajectory. This Note obtains the complete solution of this case and presents the isochrones in the (ψ_0, x_t) space. The quadratic equation[3,4] with coefficients in terms of the state variables and the integrals of motion is again used for determining the optimal bank angle.

Problem Formulation

The goemetry of gliding flight in a horizontal plane is depicted in Fig. 1a. At the initial instant, the relative geometry of the target and the vehicle can be specified by two parameters, the initial distance x_t and the initial velocity yaw angle ψ_0. The dimensionless equations of motion are[3,4]

$$x' = u\cos\psi$$

$$y' = u\sin\psi$$

$$u' = -(u^2/2E^*\omega)[1+(\omega^2/u^4\cos^2\mu)]$$

$$\psi' = -\tan\mu/u$$

$$\omega = \lambda u^2 \cos\mu \tag{1}$$

where the prime denotes a derivative taken with respect to θ and the last equation is a constraint for horizontal flight.

In order to use the maximum principle, we introduce the adjoint vector $\boldsymbol{p}$ to form the Hamiltonian

$$H = p_x u\cos\psi + p_y u\sin\psi - p_u(u^2/2E^*\omega)[1+(\omega^2/u^4\cos^2\mu)] - p_\psi(\tan\mu/u) \tag{2}$$

The maximum endurance variational problem has the integrals[5]

$$H=-1$$

$$p_x = C_1$$

$$p_y = C_2$$

$$p_\psi = C_1 y - C_2 x + C_3 \tag{3}$$

For interior (i.e., unconstrained) bank control, we have $\partial H/\partial\mu = 0$ or

$$\tan\mu = -\left(\frac{p_\psi}{p_u}\right)\left(\frac{E^* u}{\omega}\right) \tag{4}$$

The quadratic equation for the optimal bank angle can be obtained from Eqs. (2), (3), and (4)[3-5]

$$(C_1 y - C_2 x + C_3)\tan^2\mu - 2u[1+u(C_1\cos\psi + C_2\sin\psi)]\tan\mu - (C_1 y - C_2 x + C_3)(1+u^4/\omega^2) = 0 \tag{5}$$

Isochrones for Maximum Endurance Flight

To construct the isochronous lines, the maximum flight time must be determined for each pair of (ψ_0, x_t) where ψ_0 can be any value between 0 and 180 deg (or 0 and -180 deg) and x_t falls between 0 and the maximum reachable distance determined in Ref. 4. Hence the initial and final conditions are

$$\theta_0 = 0, x_0 = y_0 = 0, u_0 = 1, \psi_0 = 0 \text{ to } 180 \text{ deg}$$

$$x_f = x_t, y_f = 0, u_f = \text{specified}, \psi_f = \text{free} \tag{6}$$

and θ_f is to be maximized. Only the trajectories with ψ_0 from 0 to 180 deg, must be solved for, due to the property of symmetry. Since ψ_f is free, we have $p_\psi(\theta_f) = 0$ or

$$C_3 = C_2 x_t \tag{7}$$

Equation (5) then becomes

$$[C_1 y + C_2(x_t - x)]\tan^2\mu - 2u[1+u(C_1\cos\psi + C_2\sin\psi)]\tan\mu - [C_1 y + C_2(x_t - x)](1+u^4/\omega^2) = 0 \tag{8}$$

For numerical computation, a small lifting vehicle with the following characteristic values is used, as in Refs. 3 and 4

$$E^* = 20, \quad \lambda_{max} = 1.8, \quad \omega = 0.23, \quad n_{max} = 5 \tag{9}$$

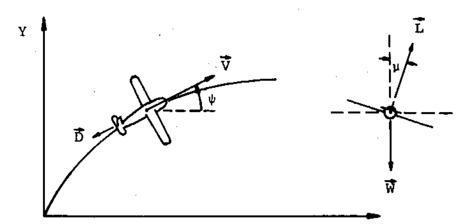

Fig. 1a Geometry of gliding flight in horizontal plane.

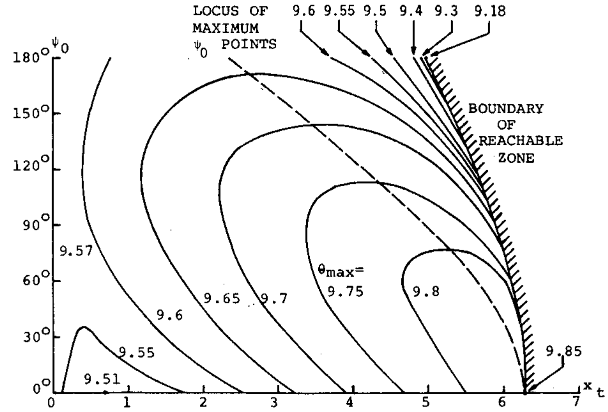

Fig. 1b Isochrones for maximum endurance flight.

The final speed is specified to be the stall speed at zero bank angle, and hence $u_f = 0.3575$. The isochrones are plotted in Fig. 1b. In some trajectories, the optimal bank angle reaches its upper bound of 78.5 deg, which is enforced by the maximum load factor constraint $n_{max} = 5$. It is seen that near the upper right corner of Fig. 1b, where both ψ_0 and x_t are large, the maximum endurance is penalized rather substantially.

Conclusion

This Note solves the most general form of the problem of maximum endurance gliding flight in a horizontal plane with the isochrones presented. The penalty on the maximum endurance is made evident when both the initial velocity yaw angle and the target distance are large.

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Stability Analysis of Gyroscopic Systems by Matrix Methods

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I. Introduction

THE application of matrix methods to a stability analysis of multiple-degree-of-freedom gyroscopic systems often provides insight into the interaction of the design parameters which otherwise may be overlooked. This additional behavioral information may in turn lead to a more informed selection of critical parameter values.

As an example of a useful matrix stability requirement, a system is said to be stable if its stiffness matrix is positive definite. Therefore, by requiring some generic stiffnes matrix to be positive definite, a set of linear inequalities in the system parameters is provided which aids in the selection process. In the case of systems described by gyroscopic forces, there are other combinations of the coefficient matrices which lead to useful stability results.

The intent of this work is to illustrate some simple stability conditions in matrix form, and, by way of example, to apply these requirements to the linear equations of motion of a dynamically tuned gyroscope (DTG). Finally, the derived regions of DTG stability are compared with those reported in the literature.[1]

In the next section, a matrix description of gyroscopic systems is given.

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II. Matrix Description of Gyroscopic Systems

The gyroscopic systems of interest here are modeled by the matrix equation

$$M\ddot{x} + G\dot{x} + Kx = 0 \tag{1}$$

where $x = x(t)$ is an $n \times 1$ displacement vector, $\dot{x}$ an $n \times 1$ vector of velocities, $\ddot{x}$ an $n \times 1$ acceleration vector, M an $n \times n$ symmetric and positive definite matrix of mass and inertia parameters, K an $n \times n$ symmetric stiffness matrix, and G an $n \times n$ skew-symmetric matrix of gyroscopic force terms.

Equation (1) is used to describe the linear vibrations of many undamped gyroscopic systems in rotating reference frames. When $n = 2$ this relationship depicts the equation of motion of a simple gyroscope.

The problem addressed here is to determine conditions on the elements of matrices M, G, and K which, when satisfied, insures a stable system performance. Matrix methods for stability analysis are discussed in the following section.

III. Stability by Matrix Methods

There are several well-known stability results available for Eq. (1). For example, if K is positive definite then the equilibrium position of Eq. (1) is stable. However, the system is not necessarily unstable if K is negative definite or indefinite (cf., Ref. 2).

To examine some of these possibilities it is convenient first to initially transform Eq. (1) into the equivalent form given by

$$I\ddot{y} + \tilde{G}\dot{y} + \tilde{K}y = 0 \tag{2}$$

where I is the $n \times n$ identity matrix, $y = M^{1/2}x$,

$$\tilde{G} = M^{-1/2}GM^{-1/2} = -\tilde{G}^T \quad \text{and} \quad \tilde{K} = M^{-1/2}KM^{-1/2} = \tilde{K}^T$$

Here the superscript T denotes the transpose of a matrix and the superscript $-1/2$ indicates the inverse of the positive definite square root of a positive definite matrix. Hagedorn[3] has shown that if the matrix $4\tilde{K} - \tilde{G}^2$ is negative definite, Eq. (2) and, hence, Eq. (1) is unstable.

A new stability result is available for a two-degree-of-freedom ($n = 2$) system. Namely, if the determinant of $\tilde{K}$ is positive and if the trace of $4\tilde{K} - \tilde{G}^2$ is positive, then Eq. (2) and, hence, Eq. (1) is stable. This can also be stated in the original coordinates. Namely, if K is negative definite with a positive determinant and if the trace of $M^{-1}(4K - GM^{-1}G)$ is positive then Eq. (1) is stable.

To verify this conjecture for the $n = 2$ case, consider the eigenvalue problem associated with Eq. (2). The characteristic equation is given by

$$\det(\lambda^2 I + \lambda\tilde{G} + \tilde{K}) = 0 \tag{3}$$

where λ is the eigenvalue and $\det(\cdot)$ denotes the determinant. Since $\tilde{K}$ is a real symmetric matrix, there exists a nonsingular orthogonal matrix S such that $S^TS = I$ and $S^T\tilde{K}S = \Lambda$, a diagonal matrix. Pre- and postmultiplying Eq. (3) by S^T and S, respectively, then yields

$$\det(\lambda^2 I + \lambda S^T\tilde{G}S + \lambda) = 0 \tag{4}$$